Periods and rounding in integer optimization

MANTIS DISCOVERY

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01

The question

Minimize a homogeneous polynomial f of degree $d \geq 2$ with rational coefficients over the nonnegative integer vectors z with $Az = nb$, and call the minimum $m_f(n)$. If v minimizes f over the real relaxation, the real minimum at scale n is exactly $n^d f(v)$. We ask when $m_f(n)$ eventually agrees with a polynomial in n on each residue class, what the correction looks like when it does not, and how often rounding the real optimum to a nearby integer point is exact.

The same question arises in experimental design, where a design has to be realised in whole observations and the natural candidate is the real optimum rounded off. Whether that rounding is right, and how the answer depends on the scale, is the practical form of the question.

02

How we are approaching it

- We show that integer optima stay within bounded distance of nv when f grows quadratically away from v.
- We read the offsets as a point rotating on a torus, so the leading correction is a nearest-lattice function of that rotation.
- We use the Subspace Theorem to bound how often an algebraic rotation lands near a wall where the answer switches.
- We solve explicit families exactly, among them two-generator cubics through Pell equations.
- We extend the classification to minima that sit on a face, where the period comes from the lattice of that face.
- We verify the explicit families by exhaustive search at small scales before trusting the general argument.

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Nine classes, nine lines

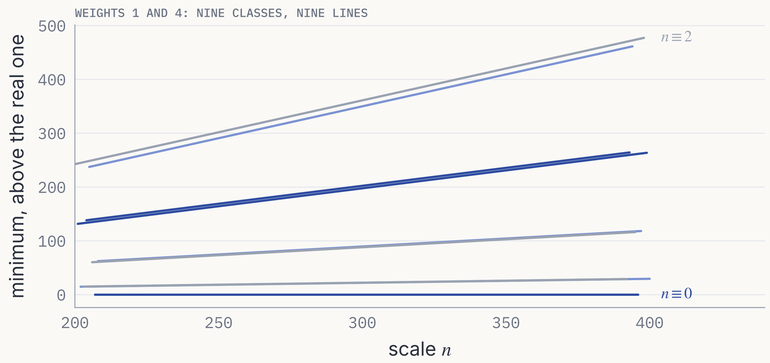


FIGURE 1. What the classification says, in one case. Minimizing $x^3 + y^3$ under $x + 4y = n$ has a rational real optimum of period 9, and the minimum then sits exactly on a straight line in n within each residue class.

04

What we have so far

Suppose the feasible set is compact, f has a unique minimizer v with every coordinate positive, and the Hessian at v is positive definite on the kernel of A. Then m_f is eventually quasipolynomial exactly when v is rational, and its least period is the smallest M with Mv integral.

Under the same hypotheses the scaled corrections converge. The limit takes finitely many values when v is rational and is a continuous law when v is irrational.

Choosing by the original cost among the minimizers of the quadratic term is exact for all $n \leq N$ with at most $O_\varepsilon(N^\varepsilon)$ exceptions, and at most $O(\log N)$ of them when the coordinates of v generate a quadratic field.

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When rounding fails

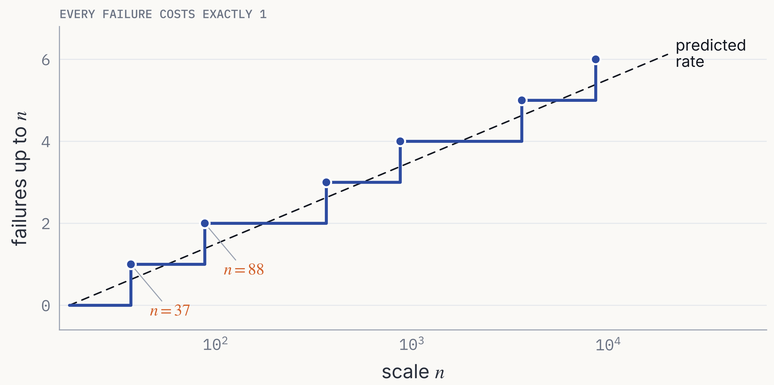


FIGURE 2. Rounding the real optimum fails rarely, and the failures are exactly the terms of two recurrences. For $x^3 + y^3$ under $2x + 3y = n$ we found every failure up to $n = 20{,}000$ by exhaustive search. Each one costs exactly 1, and the count follows the predicted logarithmic rate.

06

Open questions

- We assume one real minimizer. What happens when the minimum is attained at several points?
- Curvature cannot be dropped. We give polynomials with a unique rational minimizer whose integer minima are not eventually quasipolynomial, and we have no classification there.
- The constants are ineffective, and we have no polynomial-time algorithm for the version on a face.
- We count the exceptions exactly only for two generators. What is the sharp count in higher dimensions?