Inertia of matching cofactor matrices

MANTIS DISCOVERY

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01

The question

Take a graph G with an even number of vertices and count the perfect matchings that survive deleting two vertices:

$$W_{uv} = \mathrm{pm}(G - u - v), \qquad W_{uu} = 0.$$

How many positive, negative and zero eigenvalues does W have? One positive eigenvalue is the least possible, and it is exactly the Lorentzian property of the polynomial $\frac{1}{2} z^{\mathsf{T}} W z$ in the sense of Brändén and Huh.

02

The five graphs

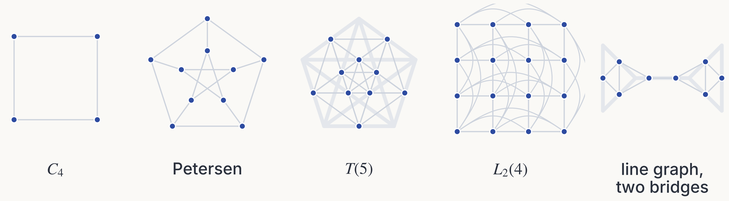


FIGURE 1. $T(5)$ and the bridged graph are line graphs, drawn over their base graphs (faint): K_5, and two triangles joined by a path of two bridges. $L_2(4)$ is the rook's graph on a 4×4 board.

03

How we are approaching it

- Compute W exactly for many small graphs and look for patterns.
- For line graphs of subcubic graphs, combine a cofactor congruence for one-vertex attachments with the parity of matchings and a signed factorization by two-edge-cut classes.
- For strongly regular graphs, use Gaussian identities and switching for the lattice and triangular families, and exact integer certificates for the rest.

04

Computed spectra

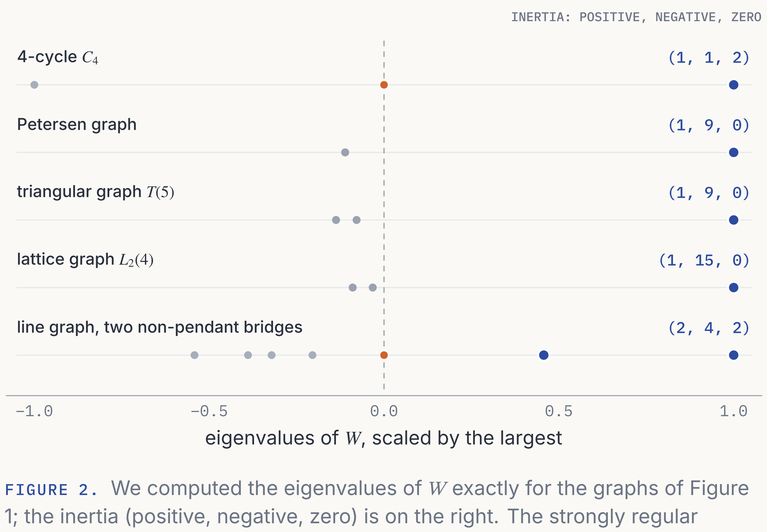


FIGURE 2. We computed the eigenvalues of W exactly for the graphs of Figure 1; the inertia (positive, negative, zero) is on the right. The strongly regular graphs have one positive eigenvalue. C_4 is the exception, with two zeros, and a base graph with two non-pendant bridges gives two positive eigenvalues.

05

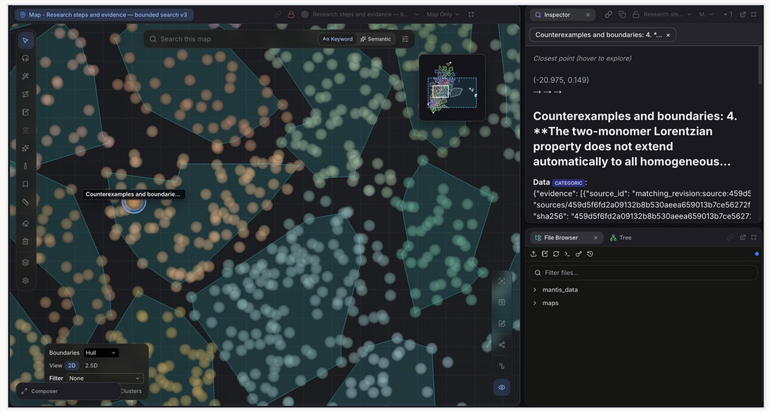
What we have so far

For line graphs of connected subcubic graphs with an even number of edges, the positive index is a count of components after a rule on bridges. W has one positive eigenvalue exactly when every bridge of the base graph is pendant.

Every connected strongly regular graph of even order with least eigenvalue -2 has inertia $(1, n-1, 0)$, except C_4, which has $(1, 1, 2)$.

06

In Mantis



3,583 RECORDS · THE OPEN RECORD

The two-monomer Lorentzian property does not extend automatically to all homogeneous polynomials.

The record gives an example: for an eight-vertex cubic base graph, the four-monomer polynomial is not Lorentzian.

ALSO ON RECORD

An invalid proof step assumed that homogeneous components of real-stable polynomials are real-stable. They need not be: $z^2 - 1$ is real-stable, but $z^2 - t^2$ vanishes at $z = t = i$.

07

Open questions

- Which deletion-count matrices are Lorentzian, and what makes the others fail?
- Line graphs with least eigenvalue -2 can have an arbitrarily large positive index. What controls it in general?